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ESTIMATING THE CORRELATION DIMENSION AND THE LARGEST LYAPUNOV EXPONENT FROM THE FGGE IIIb DATA

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Abstract

The 500 mb geopotential heights for both winter and summer from the FGGE IIIb data have been analyzed for the existence of strange attractors. Five latitudes, 60°N, 30°N, 0°(EQ), 30°S and 60°S, are selected for the present study. The correlation dimension and the largest Lyapunov exponent are estimated to obtain geometrical and dynamical characteristics of strange attractors. The correlation dimension provides a lower bound of the number of independent variables necessary to model the dynamics. The largest Lyapunov exponent provides the average exponential growth rate of nearby trajectories on an attractor in an appropriate phase space, then this is related to the predictability of the temporal development of the time series. The inverse value of the exponent gives a mean predictability time scale. The correlation dimension D_2 and the largest Lyapunov exponent Λ are estimated as follows: $D_2 \approx 9-12$ and $\Lambda \approx 0.01\,\mathrm{day}^{-1}$ at EQ, $D_2 \approx 6-9$ and $\Lambda \approx 0.02-0.04\,\mathrm{day}^{-1}$ at 30°, and $D_2 \approx 4-6$ and $\Lambda \approx 0.04-0.08\,\mathrm{day}^{-1}$ at 60°. These results show latitudinal and seasonal change.

1. Introduction

Lorenz [1] reported that solutions of a set of three ordinary differential equations modeling thermal convective motion of fluid layer, exhibit chaotic behavior. After the work of Lorenz [1], especially in the last 10 years, chaotic phenomenon has been discovered among many other fields, and theoretical study has being developed.

In chaotic phenomena the temporal development of a trajectory is

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deterministic, but it is unstable and does not approach any periodic or quasi-periodic states asymptotically. However, the trajectory is bounded on an attractor, the so-called strange attractor which has a fractal structure.

Many methods to analyze chaotic behaviors have been developed. Grassberger and Procaccia [2] introduced the correlation dimension as a useful measure of strange attractors. The correlation dimension presents the minimum number of essential variables to model the dynamics of the system. The application to meteorological data has been developed, and several numerical estimates suggested a low dimensional strange attractor in a variety of systems [3, 4, 5, 6, 7, 8]. These estimates were obtained from a data at a particular place. The purpose in the present paper is to understand the chaotic features of the atmospheric motions globally. For this purpose we use the First GARP (Global Atmospheric Research Program) Global Experiment (FGGE) IIIb data set which provide the global atmospheric data. The analyses in the present study are based on the time series of the 500 mb geopotential height.

Strange attractors can be characterized by dimensions, Lyapunov exponents, the Kolmogorov entropy, etc. [9]. In the present paper we estimate the correlation dimension and the largest Lyapunov exponent. The correlation dimension which obtained from the correlations between random points on the attractor, characterizes the geometrical aspects of the strange attractor. The largest Lyapunov exponent is the average rates of exponential divergence or convergence of nearby orbits in phase space, that provides a quantitative measure of predictability.

2. Calculation procedures

2.1 Correlation dimension

The geometrical aspect of strange attractors is characterized by generalized dimensions D_q, $q > 0$ [10]. Generally $D_q > D_{q'}$ for any $q' > q$. The dimensions provide estimates of the number of independent variables necessary to describe the motion on the attractor. In this study we use the 2nd order dimension D_2 which is called the correlation dimension. Grassberger and Procaccia [2] have proposed the algorithms to extract the correlation dimension from time series of a single variable. They have suggested that it is useful to characterize experimental data, and D_2 is close to D_0 and D_1 in most case.

When one study a system in m-dimensions, not all variables but a single-variable time series are obtained in experimental situation. From a time series embedded in large enough phase space we can reconstruct an equivalent phase space of original system [11]. To generate an m-dimensional vector from a discrete one-dimensional time series $\{X(t)\}$, we use time delayed values of the time series [12]

$$\vec{x}_t = \{X(t), X(t+\tau), X(t+2\tau), \cdots, X(t+(m-1)\tau)\}, \tag{1}$$

where m is called the embedding dimension, τ is an appropriate delay time. τ is sometimes chosen as the time when an autocorrelation function takes first zero or a mutual information takes first minimum (Nevertheless, we adapt $\tau = 1$ in the present work because total number of data points is small). Points $\{\vec{x}_t\}$ thus obtained would distribute randomly in m-dimensional phase space, but lie on the attractor. The correlation integral $C(r)$ is defined by counting the number of pairs of points whose distance is smaller than the prescribed threshold r. For limited data set with high autocorrelation, the correlation integral displays an anomalous shoulder which inhibits good estimate of dimension. Theiler [13] introduced a cutoff parameter $\omega > 1$ to improve the convergence of standard correlation algorithm.

$$C(r) = \frac{1}{N^2} \sum_{n=w}^{N} \sum_{i=1}^{N-n} H(r - |\vec{x}_{i+n} - \vec{x}_i|), \tag{2}$$

where N is the total number of points in phase space, H is the Heaviside function

$$H(x) = \begin{cases} 0, & \mathit{if} \ x < 0, \\ 1, & \mathit{if} \ x > 0. \end{cases} \tag{3}$$

The shoulder in the correlation integral plot almost disappears for $w = 500$.

If $C(r)$ grows with a power low as

$$C(r) \sim r^{d(m)}, \tag{4}$$

then the slope of the logarithm of the distribution, i.e., $\log C(r)$ versus $\log r$, leads to the dimension $d(m)$

$$d(m) = \frac{d\{\log C(r)\}}{d\{\log r\}}. \tag{5}$$

For a deterministic system, $d(m)$ should attain a saturated value as m increases. This value is the correlation dimension. However, if $d(m)$ does not reach a saturation value for increasing m, then the system is random.

2.2 Largest Lyapunov exponent

The dynamics on the strange attractors can be characterized by the Lyapunov exponents. The Lyapunov exponents are measure of exponential growth rates which relate to the time evolution of the distance between the nearby trajectories in phase space. One or more positive Lyapunov exponent implies the dynamics to be chaotic, and the magnitude of the exponent reflects the time scale on which system dynamics become unpredictable. It is difficult to determine the spectrum of Lyapunov exponents from experimental data. We use the practical method to extract the largest Lyapunov exponent from experimental data proposed by Sato et al. [14].

Consider two nearby points $\vec{x}_t$ and $\vec{y}_t$ on a time series $\{\vec{x}_t\}$ defined by (1). The time interval between $\vec{x}_t$ and $\vec{y}_t$ is greater than five days. If we define distance $|\vec{x}_t - \vec{y}_t|$ as $dis(\vec{x}_t, \vec{y}_t)$, then the distance of the trajectories at time $t + \tau$ can be written by $dis(\vec{x}_{t+\tau}, \vec{y}_{t+\tau})$. The exponential growth rate of the distance of the trajectories, i.e., the largest Lyapunov exponent Λ is given by

$$dis(\vec{x}_{t+\tau}, \vec{y}_{t+\tau}) \sim e^{\Lambda\tau}\, dis(\vec{x}_t, \vec{y}_t), \tag{6}$$

thus we obtain Λ

$$\Lambda(\tau) = \frac{1}{\tau}\left\langle \log \frac{dis(\vec{x}_{t+\tau}, \vec{y}_{t+\tau})}{dis(\vec{x}_t, \vec{y}_t)} \right\rangle, \tag{7}$$

where $\langle \cdots \rangle$ indicates average for all $\vec{x}_t$.

3. Application to the FGGE IIIb data

3.1 Data

We analyze the First GARP Global Experiment (FGGE) IIIb data produced at the Geophysical Fluid Dynamics Laboratory (GFDL). These data have twice daily values from December, 1978 to November, 1979 in a horizontal resolution of 1.875° in both latitude and longitude.

The 500 mb geopotential height at five latitudes (60°N, 30°N, 0°(EQ), 30°S and 60°S) are employed as the time series. The 500 mb geopotential height may be inappropriate for analysis of equator, and insufficient in number because the dynamical behaviors at the equator would be more complex than these at the higher latitude. In any case the behavior represented by the 500 mb geopotential height are analyzed to indicate its latitudinal change in the present paper. The analyzed periods are Northern winter (December to February) and Northern summer (June to August).

The FGGE IIIb data have 192 grid points along a latitude, we divide them 16 groups, then each group has 12 grid points which have time series individually. A time series of a group is created by connecting twelve time series to make a long one, it has 2160 values in Northern winter and 2208 values in Northern summer.

3.2 Correlation dimension

The correlation integral $C_i(r)$ is calculated each 16 groups, and add together, then the correlation dimension is estimated from slopes in a $\log \sum_{i=1}^{16} C_i(r)$ versus $\log r$ diagram as calculated from (5). Fig. 1 shows slopes versus $\log C(r)$ at 60°N, and the correlation dimension is obtained as the saturation value over a plateau corresponding to the scaling region. The correlation dimension D_2 we obtained for both Northern winter and Northern summer are listed in Table 1.

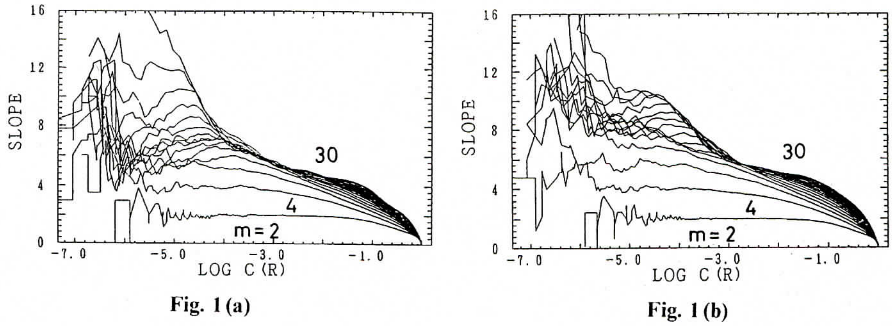

Fig. 1. Slope versus $\log C(r)$ estimated from the time series of the 500 mb geopotential height at 60°N. (a) winter, (b) summer. Embedding dimensions are $m = 2, 4, \ldots, 30$.

Table 1. Estimates of the correlation dimension D_2 from the time series of the 500 mb geopotential height. Season is represented by that in northern hemisphere.

season	60°N	30°N	0°	30°S	60°S
winter	4.6 ~ 5.2	~ 7.0	9.4 ~ 9.8	~ 8.8	~ 4.6
summer	5.0 ~ 5.7	8.0 ~ 9.0	11.4 ~ 12	~ 6.4	~ 4.6

The correlation dimension D_2 tends to grow from higher latitude to lower latitude. The next integer above the correlation dimension provides the minimum number of independent variables necessary to model the dynamics. Thus atmospheric motions are more complicated in lower latitude than in higher latitude in this sense. This result seems appropriate because in middle latitudes a set of equations can be simplified due to geostrophy, but not in low latitudes. Seasonal variation of the estimates is small in 60° in both hemispheres, while that in lower latitude seasonal variation is relatively clear in both hemispheres. The correlation dimension in summer is larger than that in winter except at 60°S (there is no difference). Atmospheric motions are more complicated in summer than in winter, and the effect of seasonal change is larger in lower latitude than in higher latitude.

Fraedrich [4] showed that the attractor dimension in summer (≈ 3.9) is larger than that in winter (≈ 3.2) for daily surface pressure records at Berlin, while he [5] obtained the correlation dimension 6.8 to 7.1 for the same time series in winter. We do not know the reason of change of the dimension. The estimates of fractal dimensions were about 8.0 using the daily 500 mb geopotential height over western Europe in Keppenne and Nicolis [7]. Zeng et al. [8] were not able to obtain saturated values for the correlation dimensions from the daily surface temperature and pressure in the United States and the North Atlantic Ocean, and concluded that the correlation dimension was greater than 8. Estimated values in our study are relatively low, and show small increase of dimension from winter to summer in 60°N.

3.3 Largest Lyapunov exponent

The largest Lyapunov exponent is calculated each 16 groups, and an averaged value from them is estimated. The behavior of the largest Lyapunov exponent versus evolution time with increasing embedding dimension are shown in Fig. 2 as calculated from (7). In this calculation the largest Lyapunov exponent decreases with increasing embedding dimension. We expect that the embedding dimension around $2D_2 + 1$ is appropriate for estimating the largest Lyapunov exponent. This is successfully tested using finite time series from the Lorenz system similar in size to the present study's time series.

Table 2 shows the results in each latitude for both seasons. The inverse value Λ^{-1}, multiplied by $\ln 2 \sim 0.69$, defines the error doubling time T. It is regarded as a mean time scale of the predictability on the attractors.

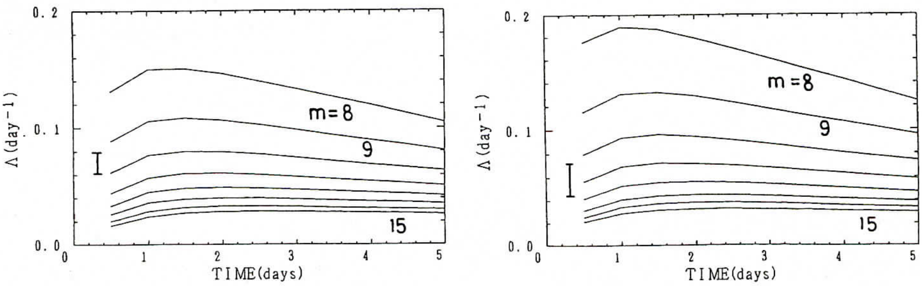

Fig. 2 (a) **Fig. 2 (b)**

Fig. 2. The largest Lyapunov exponent Λ versus time estimated from the time series of the 500 mb geopotential height at 60°N. (a) winter, (b) summer. Embedding dimensions are $m = 8, \ldots, 15$. The vertical bar indicates the fitted region.

Table 2. Estimates of the largest Lyapunov exponent Λ, and its inverse Λ^{-1}, multiplied by ln 2, gives the predictability (error doubling) time T from the time series of the 500 mb geopotential height. Season is represented by that in northern hemisphere.

season		60°N	30°N	0°	30°S	60°S
winter	Λ (day^{-1})	0.062 ~ 0.080	~ 0.035	~ 0.013	0.020 ~ 0.023	~ 0.039
	T (days)	8.7 ~ 11	~ 20	~ 53	30 ~ 35	~ 18
summer	Λ (day^{-1})	0.045 ~ 0.072	0.027 ~ 0.035	~ 0.011	~ 0.038	~ 0.038
	T (days)	9.6 ~ 15	20 ~ 26	~ 63	~ 18	~ 18

The predictability time scale in higher latitude is shorter than that in lower latitude except 60°S in summer, and the time scale in winter is shorter than that in summer except 60°S. There is little latitudinal change in middle latitudes of southern hemisphere in summer, and there is almost no seasonal change at 60°S. These results may reflect the effect of topography to the atmospheric motions.

The error doubling time is about a few days deduced from the general circulation model [15, 16, 17], while study of analogues represented that the error doubling time is about 8 days [18, 19]. Fraedrich [5] estimated the error doubling time of about 9 days using daily surface pressure values at Berlin for winter seasons. Using the 500 mb geopotential height over western Europe, Keppenne and Nicolis [7] obtained the error doubling time of about 19 days. The estimates of the error doubling time are from about 2 to 8 days for several regions of the United States and the North Atlantic Ocean [8]. Estimated error doubling time at 60°N in the present paper, which is about 10 days, seems to be comparable.

4. Conclusions

The analyses for the existence of strange attractors are performed using the time series of the 500 mb geopotential height of the FGGE IIIb data. Five latitudes, 60°N, 30°N, 0°(EQ), 30°S and 60°S, are selected for the present study. Period of the data set is winter and summer season. The correlation dimension and the largest Lyapunov exponent are estimated to obtain geometrical and dynamical characteristic of the atmospheric motions for each latitude and season. The correlation dimension provides a lower bound of the number of independent variables necessary to model the dynamics. The largest Lyapunov exponent provides a dynamical measure of strange attractors by estimating the mean rate of divergence of the distance between initially neighboring trajectories. Its inverse value defines the mean predictable time scale. The correlation dimension tends to grow from higher latitude to lower latitude, and the dimension in winter is smaller than that in summer. The largest Lyapunov exponent decreases from higher latitude to lower latitude, and the exponent in winter is larger than that in summer except at 60°S. Seasonal and latitudinal variation of these values seems to correspond with the nature of general atmospheric motions.

Smith [20] presented that the number of points N_{min} required to estimate the correlation dimension within 5 % of its true values is $N_{min} \geq 42^m$, where m is the embedding dimension. The dimension D is obtained if m satisfies $m \geq 2D + 1$. This is severe limit. More weak limit was proposed that the correlation dimension is necessary to be below $2 \log N$, and claimed if the estimated dimension is above or close to requirement, the estimate is spurious [21, 22]. In this work some estimates especially at lower latitude may exceed this requirement. Therefore these estimates should be interpreted with care. Lorenz [23] suggested that the atmosphere is so complex, but the estimated values of the dimension have seemed surprisingly low, this is regarded that if the variable selected for analysis is strongly coupled to only a few variables of the system, the estimated dimension will be considerably low. The latitudinal and seasonal variation of the present analyses indicates the change of the chaotic behaviors of the atmospheric motions represented by the 500 mb geopotential height.

In the present paper, we used one observation time unit (0.5 days) as a delay time of (1). If we take longer delay time, e.g., we investigated until $\tau = 6$ (3 days), then the correlation dimension gives lower value, i.e., less than or about 20 %.

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